

Title: Field equations in cylindrical and spherical coordinates

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Date: December 26, 2024

In this post, we provide a concise summary of the field equations of solid mechanics as they pertain to cylindrical and spherical coordinates. These representations are particularly beneficial in deriving the differential equations of axisymmetric objects and obtaining analytical solutions for displacements and stresses.

1 Cylindrical coordinates

1.1 Cauchy's equation of motion

$$\rho \dot{\mathbf{v}} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} \quad (1)$$

$$\begin{aligned} \rho \dot{v}_r &= \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r} (\sigma_{rr} - \sigma_{\theta\theta}) + \rho b_r \\ \rho \dot{v}_\theta &= \frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r} \sigma_{\theta r} + \rho b_\theta \\ \rho \dot{v}_z &= \frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r} \sigma_{zr} + \rho b_z \end{aligned} \quad (2)$$

1.2 Time derivative of vector field $\mathbf{v}$

$$\begin{aligned} \dot{v}_r &= \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{1}{r} v_\theta^2 \\ \dot{v}_\theta &= \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} - \frac{1}{r} v_r v_\theta \\ \dot{v}_z &= \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \end{aligned} \quad (3)$$

1.3 Velocity gradient

$$\mathbf{L} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r} \frac{v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r} \frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix} \quad (4)$$

2 Spherical coordinates

2.1 Cauchy's equation of motion

$$\rho \dot{\mathbf{v}} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} \quad (5)$$

$$\begin{aligned} \rho \dot{v}_r &= \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{1}{r} (2\sigma_{r\theta} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta} \cot \theta) + \rho b_r \\ \rho \dot{v}_\theta &= \frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{r} (3\sigma_{r\theta} + (\sigma_{\theta\theta} - \sigma_{\phi\phi}) \cot \theta) + \rho b_\theta \\ \rho \dot{v}_\phi &= \frac{\partial \sigma_{\phi r}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\phi\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r} (3\sigma_{r\phi} + 2\sigma_{\theta\phi} \cot \theta) + \rho b_\phi \end{aligned} \quad (6)$$

2.2 Time derivative of vector field $\mathbf{v}$

$$\begin{aligned}
\dot{v}_r &= \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\phi^2 - v_\theta^2}{r} \\
\dot{v}_\theta &= \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2}{r} \cot \theta \\
\dot{v}_\phi &= \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi}{r} \cot \theta
\end{aligned} \tag{7}$$

2.3 Velocity gradient

$$\mathbf{L} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} - \frac{v_\phi}{r} \cot \theta \\ \frac{\partial v_\phi}{\partial r} & \frac{1}{r} \frac{\partial v_\phi}{\partial \theta} & \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\theta}{r} \cot \theta + \frac{v_r}{r} \end{bmatrix} \tag{8}$$