# Analytical solutions - axisymmetric plate

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## 0.1 Thin wall plate

### 0.1.1 Introduction

Consider the axisymmetric plate geometry depicted in the figure below. The loading and boundary conditions are assumed to exhibit rotational symmetry.

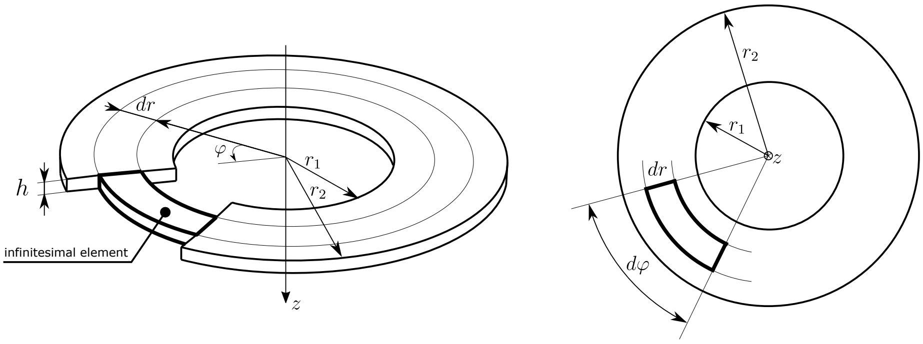


Figure 1: Geometry of the system under consideration. The small cut indicates an infinitesimally small element used for deriving the force equilibrium.

The geometry is characterized by the inner radius $r_1$, outer radius $r_2$, and thickness $h$, with the condition that $h \ll r_2$. Forces or pressure acting in the $z$ direction are presumed, adhering to the rotational symmetry. Given the prescribed loading, boundary conditions, and material properties, our objective is to solve the equation of mechanical equilibrium $\vec{\nabla} \cdot \boldsymbol{\sigma} + \vec{b}$ and determine the displacement field $\vec{u}$, as well as the stress and strain tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$. To accomplish this analytically, certain additional assumptions must be established.

Due to the small thickness of the plate, it is not possible to capture the distribution of the stress component $\sigma_r$ along the $z$ direction, as we are assuming a 2D domain passing through the middle of the plate. While the net force is zero, the bending effect caused by tensile and compressive stresses can be effectively modeled using a moment $m_r$ per unit length. This assumption can be mathematically expressed by the following equation:

$$m_r = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_r 1dz \tag{1}$$

Similarly, we can write for the moment $m_\theta$:

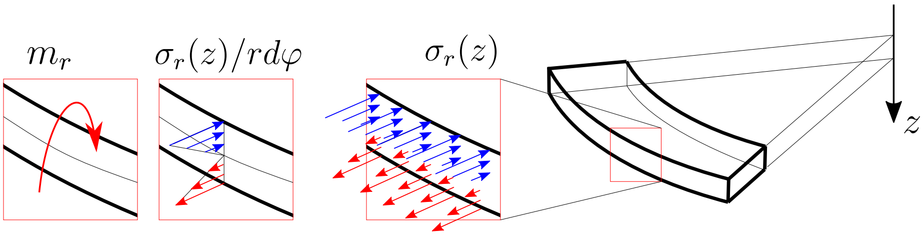


Figure 2: Visualization of the moment $m_r$ and its relationship to stress distribution: (right) Stress distribution along the circumferential cut, (middle) Stress per unit length, and (left) Equivalent representation by a moment.

$$m_\theta = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_\theta 1 dz \tag{2}$$

In addition, we will introduce the definition of the force $T$, which acts in the $z$ direction and is applied to any cross section per unit length.

#### 0.1.2 Equilbrium equations

Now, let's establish the equations of mechanical equilibrium. We derive these equations by summing up all the forces and moments acting on the infinitesimal element, as illustrated in figure below. It is important to note that these equations must hold for every infinitesimal element.

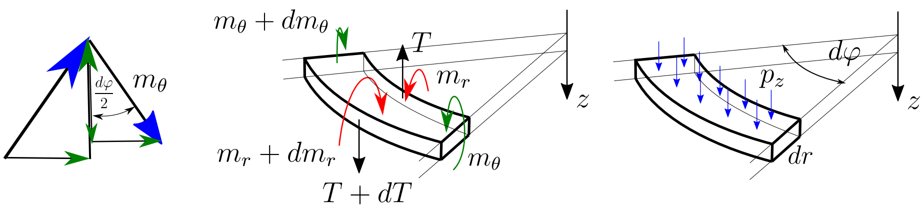


Figure 3: Infinitesimally small element.

Since the element is infinitesimally small, the increments in moment and force will also be infinitesimally small. Additionally, we will assume the presence of pressure $p_z$ as depicted in the figure.

Now, let's sum up all the forces and moments acting on the element.

$$\begin{aligned} \sum F_z: &\quad (T + dT)\,(r + dr)\,d\varphi - Trd\varphi + p_z(r)rdrd\varphi = 0 \\ \sum M_\theta: &\quad (m_r + dm_r)\,(r + dr)\,d\varphi - m_r rd\varphi - m_\theta d\varphi dr - Tdr = 0 \end{aligned}. \tag{3}$$

In the force equation, we have used $rd\varphi$ as the length of the element edge at $r$, $(r + dr)d\varphi$ as the length of the element edge at $r + dr$, and $rdr \times d\varphi$ as the area of the element upon which the pressure acts.

Regarding the momentum equation, it is important to observe how $m_\theta$ partially bends the element in the same manner as $m_r$. This component can be readily derived from the left portion of the figure above and thus become $2m_\theta \sin\frac{d\varphi}{2}$. This component of $m_\theta$, can be approximated as $2m_\theta\frac{d\varphi}{2}$ by assuming $sin(d\varphi) \approx d\varphi$. It is important to note that the other two components will cancel each other out. Furthermore, the forces $T$ will also create a moment of $Tdr$.

By simplifying these equations and neglecting second-order differentials, we obtain the following expressions:

$$\begin{aligned} \sum F_z :\ & \frac{dT}{dr} + \frac{T}{r} + p(r) = 0 \\ \sum M_\theta :\ & m_r - m_\theta + \frac{dm_r}{dr} - Tr = 0 \end{aligned} \tag{4}$$

At this point, no specific consideration of the material properties has been taken into account. Therefore, these equations hold for any material, whether it is elastic, plastic, or exhibits other non-linear behaviour.

### 0.1.3 Element deformation

Next, we will establish the deformation of the plate. The figure below illustrates the undeformed and deformed configurations.

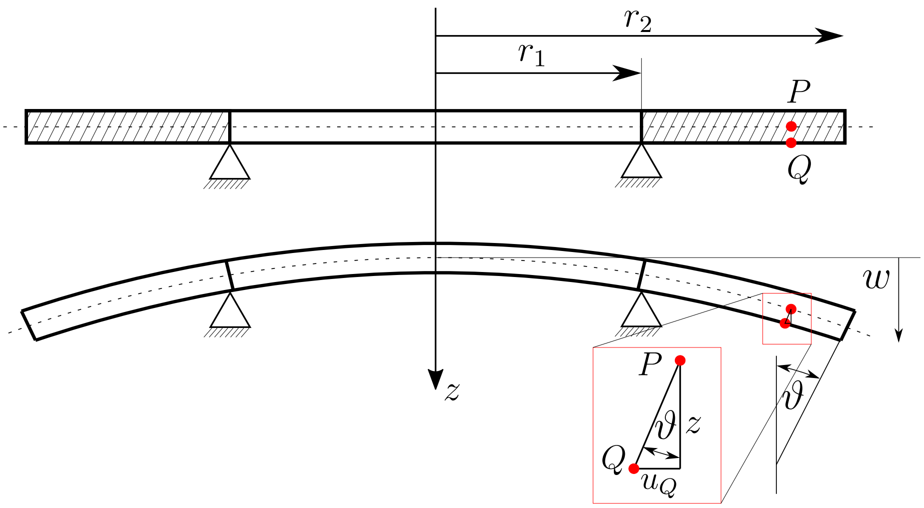


Figure 4: Undeformed and deformed configurations of the plate.

The strain tensor can be expressed in the following form, with the shear components assumed to be zero due to the axisymmetric assumptions that prohibit any shear deformation:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_r & 0 & 0 \\ 0 & \varepsilon_\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5}$$

In polar coordinates, it can be shown that $\varepsilon_r = \frac{du}{dr}$, where $u$ represents the horizontal displacement component. Additionally, $\varepsilon_\theta = \frac{1}{r}\frac{d\theta}{d\varphi} + \frac{u}{r}$, but due to the axisymmetry, the first term will vanish, resulting in $\varepsilon_\theta = \frac{u}{r}$.

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \frac{du}{dr} & 0 & 0 \\ 0 & \frac{u}{r} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

By examining the positions of the P and Q points before and after deformation, as depicted in the subfigure, we can establish a relationship between horizontal displacement and rotation:

$$\sin\vartheta = \frac{-u_Q}{z} \tag{7}$$

From this equation, we can deduce that $u_Q = -z\sin\vartheta$. For small rotations, we can approximate $\sin\vartheta$ as $\vartheta$, resulting in $u_Q = -z\vartheta$. It is important to note that this relationship holds for any point on the body.

The strain tensor finally becomes:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} -z\frac{d\vartheta}{dr} & 0 & 0 \\ 0 & -z\frac{\vartheta}{r} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

### 0.1.4 Constitutive model

After deriving the equations of mechanical equilibrium and the strain tensor, we can establish the relationship between stress and strain using a constitutive model. In this case, we will assume a linear elastic material response, which follows Hooke's Law:

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} \tag{9}$$

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_\theta\right) \\ \sigma_\theta &= \frac{E}{1-\nu^2}\left(\varepsilon_\theta + \nu\varepsilon_r\right) \end{aligned} \tag{10}$$

### 0.1.5 Differential equation

We can now proceed to calculate the moments $m_r$ and $m_\theta$ using the equations derived earlier:

$$m_r = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_r dz = -\frac{E}{1-\nu^2}\left(\frac{d\vartheta}{dr} - \nu\frac{\vartheta}{r}\right)\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 dz = -\underbrace{\frac{Eh^3}{12(1-\nu^2)}}_{=B}\left(\frac{d\vartheta}{dr} - \nu\frac{\vartheta}{r}\right) \tag{11}$$

where $B = \frac{Eh^3}{12(1-\nu^2)}$ is the bending stiffness.

Similarly,

$$m_\theta = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_\theta dz = -\frac{Eh^3}{12(1-\nu^2)}\left(\frac{\vartheta}{r} + \nu\frac{d\vartheta}{dr}\right) \tag{12}$$

We can now use the second equation of mechanical equilibrium to arrive to the following differential equation

$$\frac{d^2\vartheta}{dr^2} + \frac{1}{r}\frac{d\vartheta}{dr} - \frac{\vartheta}{r^2} = -\frac{T(r)}{B} \tag{13}$$

which can be also written using $w$:

$$\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr} = -\frac{T(r)}{B} \tag{14}$$

## 0.2 Solutions to specific cases

Here, we present the solutions to the derived differential equations for specific boundary conditions and loading cases. The differential equation can also be expressed in the following format:

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\vartheta)\right] = -\frac{T(r)}{B} \tag{15}$$

The solution for the rotation field $\vartheta$ can be found as

$$\vartheta = c_1 r + \frac{c_2}{r} - \frac{1}{Br}\int_r \left(r\int_r T(r)dr\right)dr \tag{16}$$

Similarly, the displacement component $w$ can be shown to be:

$$w = c_1\frac{r^2}{2} + c_2\ln(r) - \frac{1}{B}\int_r \left[\frac{1}{r}\int_r \left(r\int_r T(r)dr\right)dr\right]dr \tag{17}$$

For any given function $T(r)$, which can vary along $r$ and can also be derived from the pressure $p$, an analytical solution can be obtained. In the following, we will consider a couple of examples and compare them with a finite element analysis for comparison.

## 0.3 Finite element analysis of solid and comparison to thin-wall analytical solution

An important question to address is how closely the analytical solution, assuming thin-wall assumptions, approximates the solid solution. We also need to determine the cases for which the analytical solution can be reliably used. To answer these questions, we will solve the full equations using the finite element method and compare the results with the assumptions we have made, considering different ratios of $h/r$.

In order to validate the analytical solution, we will compare it with the numerical solution obtained through the finite element method for the cases presented in the previous section. By comparing the analytical and numerical results, we can assess the accuracy and applicability of the analytical solution.

### 0.3.1 Plate with $r_1 = 0$ and uniform pressure $p$

Using the first equation of mechanical equilibrium ($\sum F_z = 0$), and integrating, we can find the general solution:

$$T(r) = \frac{c_1}{r} - \frac{1}{r}\int_0^r rp(r)dr, \tag{18}$$

where $c_1$ is a constant.

If there are no forces other than the pressure acting on the plate, $c_1 = 0$. For the constant pressure, we can then easily derive

$$T(r) = -\frac{pr}{2}. \tag{19}$$

Integrating the previous equations, the rotation becomes and vertical displacement become:

$$\vartheta = \frac{pr^3}{16B}, \qquad w = \frac{pr^4}{64B} \tag{20}$$