Rotating disk

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0.1 Rotating disk

0.1.1 Introduction

Let us consider the following geometry of a rotating disk as depicted in Fig.1. The dimensions are defined by the inner radius r_1 and outer radius r_2. It is assumed that the wall thickness h is much smaller than r_2. The disk is rotating with angular velocity of ω. An infinitesimally small element is selected for the derivation of deformation and equilibrium equations.

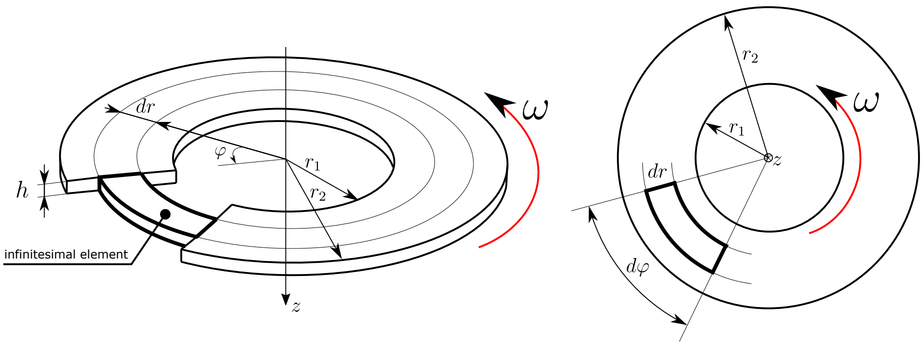

Figure 1: Geometry of a rotating disk. The small cut indicates an infinitesimally small element used for the derivation of deformation and equilibrium equations. Here, ω is the angular velocity.

0.1.2 Equilibrium equations

Equilibrium equations are derived from the forces acting on an infinitesimal element. In comparison to a thin wall, in this case, there is no presence of moments which makes the derivation of the governing equations easier. The forces acting on the infinitesimal element are presented in Fig.2.

We beging with summing up all the forces acting int the r direction.

$$\sum F_r : (\sigma_r + d\sigma_r)(r + dr)\, d\varphi h - \sigma_r r d\varphi h + dF_\omega - 2\sigma_\theta \frac{d\varphi}{2} dr h = 0 \tag{1}$$

The volume force dF_ω can be expressed through angular velocity:

$$dF_\omega = dm r\omega^2 = \rho dV r\omega^2 = \rho r d\varphi dr h r\omega^2 = \rho r^2 h\omega^2 dr d\varphi \tag{2}$$

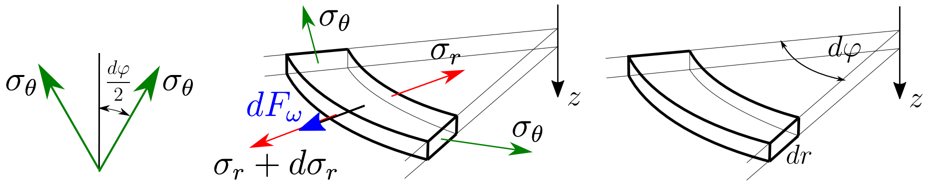

Figure 2: Forces acting on an infinitesimal element.

By simplifying the previous equations, we arrive at the following differential equation:

$$r\frac{d\sigma_r}{dr} + \sigma_r - \sigma_\theta = -\rho r^2\omega^2 \tag{3}$$

As no mention of material properties has been introduced, this equation holds for any material (elastic, plastic, etc.).

0.1.3 Element deformation

Element deformation can be defined by the following strain tensor

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_r & 0 & 0 \\ 0 & \varepsilon_\theta & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \tag{4}$$

where the individual components can be derived from analysing the deformed and undeformed shape of the element and considering the axisymmetry. This leads to:

$$\begin{aligned} \varepsilon_r &= \frac{du}{dr} \\ \varepsilon_\theta &= \frac{u}{r} \end{aligned} \tag{5}$$

0.1.4 Constitutive model

In order to connect stress and strain tensor, we will limit ourselves to the simple linear elastic material model with two material parameters, Young's modulus E and Poisson's ratio ν.

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} \tag{6}$$

THe following stress-strain relationship is then established:

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_\theta\right) \\ \sigma_\theta &= \frac{E}{1-\nu^2}\left(\varepsilon_\theta + \nu\varepsilon_r\right) \end{aligned} \tag{7}$$

0.1.5 Governing differential equation

By substituting these equations into the equation of force equilibrium, we finally arrive at:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r} = -\frac{1-\nu^2}{E}\rho r \omega^2 \tag{8}$$

0.2 Solutions

It can be shown that the following is a general solution

$$u = c_1 r + \frac{c_2}{r} - \frac{1-\nu^2}{8E}\rho r^3 \omega^2 \tag{9}$$

where c_1 and c_2 are constants to be determined for the specific case of boundary conditions. SOme of the specific cases are considered in the section below together with a finite element analysis.

0.2.1 Case with $r_1 = 0$.