Analytical solutions - cylinder

Jakub Mikula

July 31, 2023

0.1 Cylinder

0.1.1 Introduction

We will consider a geometry of cylinder depicted in Fig.1. The inner and the outer radius are denoted by r_1 and r_2, respectively. There is no constrain put on the length of the cylinder. An infinitesimally small element $dr \times d\varphi \times dh$ is selected for the derivation of deformation and equilibrium equations.

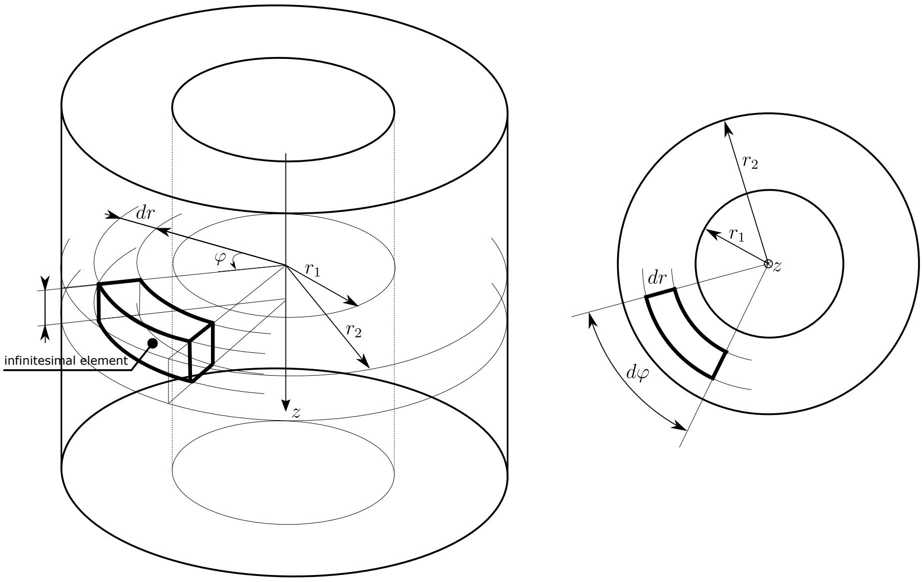

Figure 1: Geometry under consideration depicting the infinitesimally small element.

0.1.2 Equilibrium equations

Equilibrium equations are derived from the forces acting on the infinitesimally small element.

$$\sum F_r : (\sigma_r + d\sigma_r)(r + dr)\, d\varphi dz - \sigma_r r d\varphi dz - 2\sigma_\theta \frac{d\varphi}{2} dr dz = 0 \qquad (1)$$

Additionally, pressure may be applied which clearly leads to $\sigma_z = p_z$.

The equation of mechanical equilibrium in r becomes:

$$r\frac{d\sigma_r}{dr} + \sigma_r - \sigma_\theta = 0 \tag{2}$$

Notice that equilibrating the $\sum F_\varphi = 0$ will be already satisfied (leaving to reader to try).

0.1.3 Element deformation

Following the axisymmetry constrains, the strain tensor is of the following form:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_r & 0 & 0 \\ 0 & \varepsilon_\varphi & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \tag{3}$$

Analysing the element deformation, it can de derived that:

$$\begin{aligned} \varepsilon_r &= \frac{du}{dr} \\ \varepsilon_\varphi &= \frac{u}{r} \\ \varepsilon_z &= \frac{dw}{dz} \end{aligned} \tag{4}$$

Since axisyemmtry does not permit shear, all shear components of the strain tensor are zero.

0.1.4 Constitutive model

We will consider linear elastic material model with two material parameters: Young's modulus E and Poisson's ratio ν. This leads to the following stress-strain relationship:

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_\theta\right) \\ \sigma_\theta &= \frac{E}{1-\nu^2}\left(\varepsilon_\theta + \nu\varepsilon_r\right) \end{aligned} \tag{5}$$

0.1.5 Governing differential equation

By combing the derived equations, we arrive at the following governing differential equation:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \tag{6}$$

The analytical solution to this differential equation is of the following form:

$$u = c_1 r + \frac{c_2}{r} \tag{7}$$

where the constants c_1 and c_2 are determined from the boundary conditions.

0.2 Solutions