Finite element formulation of Cahn-Hilliard equation

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The most common example of the Cahn-Hilliard system is that of the separation of two immiscible phases. A simulation of such a process is illustrated below:

In this post we illustrate the finite element formulation of the Cahn-Hilliard equation.

0.1 Cahn-Hilliard equation

We will adopt the following simpler form of the equation:

$$\frac{\partial c}{\partial t} = D\boldsymbol{\nabla}^2 \left(\underbrace{c^3 - c - \gamma \boldsymbol{\nabla}^2 c}_{\mu} \right) \tag{1}$$

0.2 Finite element formulation

First, we will consider a system of two equations of two unknowns:

$$\begin{aligned} \frac{\partial c}{\partial t} &= D\boldsymbol{\nabla}^2 \mu \\ \mu &= c^3 - c - \gamma \boldsymbol{\nabla}^2 c \end{aligned} \tag{2}$$

thus solving equations where the weak form does not contain higher order derivatives of the shape functions.

We will assume the following approximation of the concentration field and the chemical potential:

$$\begin{aligned} c(t, \vec{x}) &\approx c_i(t)\varphi_i(\vec{x}) \\ \mu(\vec{x}) &\approx \mu_i \varphi_i(\vec{x}) \end{aligned} \tag{3}$$

where c_i and μ_i are the nodal values of the respective fields, and φ_i are the shape functions that we will specify later.

$$\begin{aligned} \int_{\Omega^e} \dot{c}_i \varphi_i \varphi_j d\Omega^e &= \int_{\Omega^e} \mu_i \boldsymbol{\nabla}^2 \varphi_i \varphi_j d\Omega^e \\ \int_{\Omega^e} \mu_i \varphi_i \varphi_j &= \int_{\Omega^e} (c_i \varphi_i)^3 \varphi_j d\Omega^e - \int_{\Omega^e} c_i \varphi_i \varphi_j d\Omega^e - \int_{\Omega^e} \gamma c_i \boldsymbol{\nabla}^2 \varphi_i \varphi_j d\Omega^e \end{aligned} \tag{4}$$