Voigt notation

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July 17, 2023

0.1 Notation

In this post, we will utilize specific notations to enhance clarity and understanding. To indicate the rank of a second-order tensor, we will use a subscript, such as

$$\underline{\sigma}_2$$

Vectors and matrices, which are tensors of first and second order, will be denoted in bold as $\boldsymbol{\sigma}$ For tensors of any order, we will adopt index notation, represented as

$$\sigma_{ij}$$

Throughout the discussion, we will adhere to Einstein's summation rule.

0.2 Introduction

Voigt notation offers a convenient approach to represent tensors in solid mechanics. It aims to simplify equations by employing vectors and matrices instead of higher order tensors, particularly benefiting symmetric tensors.

It is important to note that not all tensors encountered in solid mechanics are symmetric. For instance, the nominal stress $\boldsymbol{P}$ is a non-symmetric tensor, while the Cauchy stress $\boldsymbol{\sigma}$ is symmetric. In the subsequent discussion, we will solely consider symmetric tensors.

To illustrate the notation, let's examine the strain energy density ψ using an example:

$$\begin{aligned} \psi &= \underbrace{\frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\varepsilon_{ij}C_{ijkl}\varepsilon_{kl}}_{\text{index notation}} \\ \psi &= \underbrace{\frac{1}{2}\underline{\sigma}_2 : \underline{\varepsilon}_2 = \frac{1}{2}\underline{\varepsilon}_2 : \underline{C}_4 : \underline{\varepsilon}_2}_{\text{tensor notation}} \end{aligned} \tag{1}$$

In this context, σ_{ij} represents stress, ε_{ij} denotes strain, and C_{ijkl} signifies the tensor of elastic constants. The symbol ":" is used to denote the tensor contraction operation. The symmetric stress tensor can be expressed as follows:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

where $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ is a result of the momentum conservation. Therefore, there are only 6 independent components of the stress tensor. The same symmetry can be shown for the strain tensor, stemming from the conservation of angular momentum.

The above equation for the strain energy can be expanded into:

$$\psi = \sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + 2\sigma_{23}\varepsilon_{23} + 2\sigma_{21}\varepsilon_{21} + 2\sigma_{12}\varepsilon_{12}$$

It is easy to recognize that the above expansion can be conveniently written as a dot product of two vectors:

$$\psi = \underbrace{\boldsymbol{\sigma}^T\boldsymbol{\varepsilon}}_{\text{Voigt notation}}$$

where

$$\boldsymbol{\sigma}^T = \begin{bmatrix}\sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{yz} & \sigma_{zx} & \sigma_{xy}\end{bmatrix}$$

$$\begin{aligned}\boldsymbol{\varepsilon}^T &= \begin{bmatrix}\varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & 2\varepsilon_{yz} & 2\varepsilon_{zx} & 2\varepsilon_{xy}\end{bmatrix} \\ &= \begin{bmatrix}\varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \gamma_{yz} & \gamma_{zx} & \gamma_{xy}\end{bmatrix}\end{aligned} \tag{2}$$

The elements in the vector follow the following indexing convention: the first set of elements represents the diagonal elements xx, yy, and zz, followed by the shear elements in the same order with the missing index x, y, and z.

The tensor of elastic constants, represented in Voigt notation, is a 6×6 matrix:

$$\boldsymbol{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}$$

with the total of maximum 21 material constants that describe a fully anistropic material.

The strain energy density can then be written in the format:

$$\psi = \frac{1}{2}\boldsymbol{\varepsilon}^T\boldsymbol{C}\boldsymbol{\varepsilon}$$

It is important to differentiate between the tensorial shear strain components, denoted by ε_{ij}, and the engineering shear strain components, denoted by γ_{ij}. The engineering shear strains are twice the magnitude of the tensorial shear strain. Although the engineering notation is more commonly used in commercial software, it is crucial to be aware of the potential confusion that may arise.

While tensor notation is often preferred for its elegance when writing equations, Voigt notation proves useful when implementing code that utilizes standard linear algebra packages for vector-matrix operations.

0.3 Notes

Collection of notes useful for writing a code.

0.3.1 Stress tensor contraction

The contraction of the stress tensor on itself $\boldsymbol{\sigma} : \boldsymbol{\sigma} = \sigma_{ij}\sigma_{ij}$ can be writen using the Voigt notation as:

$$\boldsymbol{\sigma}^T \boldsymbol{I}_R \boldsymbol{\sigma}$$

where $\boldsymbol{I}_R$ is the Reuter's matrix:

$$\boldsymbol{I}_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

An example is the von-Mises yield surface:

$$f = \sqrt{\boldsymbol{\sigma}^T \boldsymbol{I}_R \boldsymbol{\sigma}} - \sigma_{eq} = 0$$

0.3.2 Strain tensor contraction

Similarly, to the previous example $\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} = \varepsilon_{ij} : \varepsilon_{ij}$ is in the Voigt notation expressed as

$$\boldsymbol{\varepsilon}^T \boldsymbol{I}_R^{-1} \boldsymbol{\varepsilon}$$

An example is the plastic equivalent strain:

$$\bar{\varepsilon}^p = \int_t \sqrt{\dot{\boldsymbol{\varepsilon}}^{p^T} \boldsymbol{I}_R^{-1} \dot{\boldsymbol{\varepsilon}}^p} d\tau$$