Title: Homogenized plasticity

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1 Theory

The objective of this post is the establish the basic formulations of a homogenized plasticity model. By 'homogenized' we refer to the macroscopic description of polycrystalline aggregates in which the dislocation activity on individual slip systems is averaged over a large number of randomly oriented grains. In general, this theory is not applicable to single crystals.

1.1 Rate-independent formulation

1.2 Isotropic von-Mises plasticity

In the von-Misses theory, the yield criterion is derived from the critical shear strain energy that initiates plastic flow. This leads to a yield surface that is a function of the second invariant of the deviatoric stress tensor:

$$\begin{aligned} f(\boldsymbol{\sigma}, \boldsymbol{q}) &= \sqrt{2J_2} - \sqrt{\frac{2}{3}}\left(\sigma_y + H\alpha\right) \\ &= \sqrt{\boldsymbol{s} : \boldsymbol{s}} - \sqrt{\frac{2}{3}}\left(\sigma_y + H\alpha\right) \\ &= ||\boldsymbol{s}|| - \sqrt{\frac{2}{3}}\left(\sigma_y + H\alpha\right) \end{aligned} \tag{1}$$

where the second invariant of the deviatoric stress tensor is defined as:

$$J_2 = \frac{1}{2}\boldsymbol{s} : \boldsymbol{s}, \tag{2}$$

and the deviatoric stress as $\boldsymbol{s} = \boldsymbol{\sigma} - \mathbf{1}\frac{1}{3}\mathrm{tr}\boldsymbol{\sigma}$.

1.3 Numerical integration

2 Insights into 1D homogenized plasticity

2.1 Plastic flow regime in rate-independent formulation

Consider a simple 1D problem in which the material is strained and undergoes a plastic flow. The following are the set of equations to describe this problem:
The constitutive equation:

$$\dot{\sigma} = E\left(\dot{\varepsilon} - \dot{\lambda}\frac{\partial f}{\partial \sigma}\right). \tag{3}$$

The yield surface:

$$\begin{aligned} f &= \sigma - (\sigma_y + H\alpha) = 0 \\ \dot{f} &= \dot{\sigma} - H\dot{\alpha} = 0 \end{aligned} . \tag{4}$$

The internal variable α representing the equivalent plastic strain:

$$\alpha = \bar{\varepsilon}^p = \int_0^t \dot{\varepsilon}^p d\tau = \int_0^t \dot{\lambda} d\tau. \tag{5}$$

Assuming the plastic flow only, and putting all the equations together, we get:

$$E\frac{d\varepsilon}{d\lambda}\frac{d\lambda}{dt} - E\frac{d\lambda}{dt} - H\frac{d\lambda}{dt}. \tag{6}$$

which gives the scalar plastic multiplier:

$$\lambda = \frac{E}{E+H}\varepsilon, \tag{7}$$

during the plastic flow.
Finally, the stress-strain curve becomes:

$$\sigma = \begin{cases} E\varepsilon & \text{if} \quad \varepsilon < \frac{\sigma_y}{E} \\ E\left[\varepsilon - \frac{E}{E+H}\left(\varepsilon - \frac{\sigma_y}{E}\right)\right] & \text{if} \quad \varepsilon \geq \frac{\sigma_y}{E} \end{cases}. \tag{8}$$

The two interesting cases emerge (in the plastic regime) when $H = 0$, then $\sigma = \sigma_y$ and when $H \to \infty$, then $\sigma = E\varepsilon$.
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3 Rate-dependent formulation

3.1 Plastic flow regime in rate-dependent formulation

Let's assume a simple power-law function and define the plastic strain rate as:

$$\dot{\varepsilon}^p = \dot{\gamma}_0 \left(\frac{\sigma}{s_0}\right)^{\frac{1}{m}} = \dot{\gamma}_0 \left(\frac{\sigma}{s_0}\right)^n \tag{9}$$

Substituting into a simple constitutive model, one arrives to the following:

$$\dot{\sigma} = E\left(\dot{\varepsilon} - \dot{\gamma}_0\left(\frac{\sigma}{s_0}\right)^n\right) \tag{10}$$

This represents a non-linear equation that can be discretized in time and solved using the Newton-Raphson method.

$$\frac{\sigma_{i+1} - \sigma_i}{\Delta t} = E\left(\frac{\Delta\varepsilon}{\Delta t} - \dot{\gamma}_0\left(\frac{\sigma_{i+1}}{s_0}\right)^n\right) \tag{11}$$

$$\frac{E\dot{\gamma}_0}{s_0^n}\sigma_{i+1}^n + \frac{1}{\Delta t}\sigma_{i+1} - \left(E\frac{\Delta\varepsilon}{\Delta t} + \frac{\sigma_i}{\Delta t}\right) = 0 \tag{12}$$

Using the Newton-Raphson method, we get

$$\begin{aligned} \sigma_{i+1}^{k+1} &= \sigma_{i+1}^k - \frac{f(\sigma_{i+1}^k)}{f'(\sigma_{i+1}^k)} \\ \sigma_{i+1}^{k+1} &= \sigma_{i+1}^k - \frac{\frac{E\dot{\gamma}_0}{s_0^n}\left(\sigma_{i+1}^k\right)^n + \frac{1}{\Delta t}\sigma_{i+1}^k - \left(E\frac{\Delta\varepsilon}{\Delta t} + \frac{\sigma_i}{\Delta t}\right)}{\frac{nE\dot{\gamma}_0}{s_0^n}\left(\sigma_{i+1}^k\right)^{n-1} + \frac{1}{\Delta t}} \end{aligned} \tag{13}$$