Title: Displacement Field

Subtitle: Displacement field from strain tensor

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Date: December 25, 2024

1 Displacement field from strain tensor

Given a strain tensor $\boldsymbol{\varepsilon}$, the objective is to calculate the displacement field $\boldsymbol{u}$ with respect to some fixed reference coordinate system.
Within the infinitesimal strain theory, the strain tensor is calculate from the displacement field from

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1}$$

and contains 6 independent components:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ * & \varepsilon_{yy} & \varepsilon_{yz} \\ * & * & \varepsilon_{zz} \end{bmatrix}. \tag{2}$$

The displacement field under constant strain ε^a_{ij} can be then calculated

$$\begin{aligned} u &= \varepsilon^a_{xx}x + f1(y) + f2(z) \\ v &= \varepsilon^a_{yy}y + g1(x) + g2(z) \\ w &= \varepsilon^a_{zz}z + h1(y) + h2(x) \\ \frac{\partial f1}{\partial y} + \frac{\partial g1}{\partial x} &= \varepsilon^a_{xy} \to B + D = \varepsilon^a_{xy} \\ \frac{\partial g2}{\partial z} + \frac{\partial h1}{\partial y} &= \varepsilon^a_{yz} \to F + H = \varepsilon^a_{yz} \\ \frac{\partial h2}{\partial x} + \frac{\partial f2}{\partial z} &= \varepsilon^a_{zx} \to J + L = \varepsilon^a_{zx} \end{aligned}, \tag{3}$$

from which it is obtained

$$\begin{aligned} u &= \varepsilon^a_{xx}x + A + By + K + Lz = \varepsilon^a_{xx}x + A' + By + Lz \\ v &= \varepsilon^a_{yy}y + C + Dx + E + Fz = \varepsilon^a_{yy}y + C' + Dx + Fz. \\ w &= \varepsilon^a_{zz}z + G + Hy + I + Jx = \varepsilon^a_{zz}z + G' + Hy + Jx \end{aligned} \tag{4}$$

Fixing a $a \times a \times a$ cube in space with the following boundar condtions: $u(0,0,0) = 0$, $v(0,0,0) = 0$, $w(0,0,0) = 0$, $v(a,0,0) = 0$, $w(a,0,0) = 0$, $w(a,a,0) = 0$, six equations of the six unknown constants are solved:

$$\begin{aligned} A' &= 0 \\ B &= \varepsilon^a_{xy} \\ C' &= 0 \\ D &= 0 \\ F &= \varepsilon^a_{yz}, \\ G' &= 0 \\ H &= 0 \\ J &= 0 \\ L &= \varepsilon^a_{zx} \end{aligned} \tag{5}$$

which gives:

$$
\begin{aligned}
u &= \varepsilon^a_{xx} x + \varepsilon^a_{xy} y + \varepsilon^a_{xz} z \\
v &= \varepsilon^a_{yy} y + \varepsilon^a_{yz} z \qquad\qquad . \\
w &= \varepsilon^a_{zz} z
\end{aligned}
\tag{6}
$$