Title: Isotropic elasticity

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1 Introduction

The constitutive relation (relation between stress and strain) in linear elasticity (known as the Hook's law) is given by the proportionality between stress and strain, $\sigma = \mathbf{D}\varepsilon$, where σ is a second-rank stress tensor, ε is a second-rank strain tensor and $\mathbf{D}$ is a fourth-rank tensor of elastic constants.
The simplest but most often implemented tensor of elastic constants (mainly for polycrystalline materials) is that of isotropic elasticity.

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & D_{12} & 0 & 0 & 0 \\ D_{12} & D_{11} & D_{12} & 0 & 0 & 0 \\ D_{12} & D_{12} & D_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{44} \end{bmatrix}, \tag{1}$$

in which

$$\begin{matrix} D_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \\ D_{12} = \frac{E\nu}{(1+\nu)(1-2\nu)} \\ D_{44} = \frac{E}{2(1+\nu)} \end{matrix}, \tag{2}$$

where E is the Young's modulus and μ is the Poisson's ratio. It is only these two constants sufficient for describing the isotropic behaviour.
This constitutive relation in 2D is simplified into

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{11} & 0 \\ 0 & 0 & C_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}, \tag{3}$$

where the elastic constants C_{11}, C_{12}, and C_{44} are defined for
- plane strain

$$\begin{matrix} C_{11} = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)} \\ C_{12} = \frac{E\mu}{(1+\mu)(1-2\mu)} \\ C_{44} = \frac{E}{2(1+\mu)} \end{matrix}. \tag{4}$$

- plane stress

$$\begin{matrix} C_{11} = \frac{E}{1-\mu^2} \\ C_{12} = \frac{E\mu}{1-\mu^2} \\ C_{44} = \frac{E}{2(1+\mu)} \end{matrix}. \tag{5}$$

For an elastically isotropic material the following condition holds: $C_{44} = \frac{C_{11}-C_{12}}{2}$. There is no directionality in the elastic properties. Polycrystalline materials may be considered isotropic as the sufficient amount of grains with a random (non-textured) grain orientation will average the elastic properties.